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NOETHER SYMMETRIES AND SOME EXACT SOLUTIONS IN $f(R, T^2)$ THEORY

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Abstract. The main objective of this article is to examine some physically viable solutions through the Noether symmetry technique in $f(R, T^2)$ theory. In order to investigate Noether equations, symmetry generators and conserved quantities, we use a specific model of this modified theory. We find exact solutions and examine the behavior of various cosmological quantities. It is found the behavior these quantities is consistent with current observations indicating that this theory describes the cosmic accelerated expansion. We conclude that generators of Noether symmetry and conserved quantities exist in this theory.

1. Introduction. The current cosmic expansion has been the most stunning and dazzling result for the scientific community [1]. Although general relativity (GR) is a widely accepted theory which explains the cause of this expansion but it has some issues like coincidence and fine tuning problems. To addresses these issues, several modifications of GR (modified gravitational theories) have been formulated to unveil the cosmic mysteries. The first modification of GR is $f(R)$ theory and significant literature [2] is available to understand the physical features of this theory. Recently, Katirci and Kavuk [3] modified $f(R)$ theory by introducing a non-linear term ($T^2 = T_{\xi\eta}T^{\xi\eta}$) in the functional action referred to as $f(R, T^2)$ theory.

This proposal is also dubbed as energy-momentum squared gravity (EMSG) and contains higher-order matter source terms which are helpful to analyze various interesting cosmological results. It is worthwhile to mention here that this theory explains the complete cosmic history and the cosmic evolution. Roshan and Shojai [4] examined that EMSG resolves the primordial singularity as it has bounce in the early universe. Board and Barrow [5] used a specific model of this theory and discussed exact solution, singularities as well as cosmic evolution with the isotropic configuration of matter in this theory. Bahamonde et al [6] studied various EMSG models and analyzed that these models manifest the current cosmic evolution and acceleration. We have examined some physically viable solutions [7] and dynamics of celestial objects in this theory [8].

The Noether symmetry (NS) strategy gives a fascinating method to develop new cosmic models and associated structures in modified theories of gravity. The NS approach is significant as it recovers symmetry generators as well as some conservation laws of the system [9]. This method does not deal only with the dynamical solutions but it also provides some viable conditions to select cosmic models based on recent observations [10]. Moreover, this method is an important and useful technique to examine exact solutions by using conserved values of the system. Conservation laws are the main ingredients to analyze the distinct physical phenomena. These are the particular cases of the Noether theorem, according to which every differentiable symmetry produces conservation laws. The conservation laws of linear and angular momentum govern the translational and rotational symmetry of any object. The Noether charges are important in the literature as they are used to examine various major cosmic problems in various considerations [12–21].

This manuscript investigates the NS for anisotropic and homogenous cosmic models such as BT-I, BT-III and Kantowski–Sachs (KS) in the background of EMSG. The manuscript is organized as follows. Section

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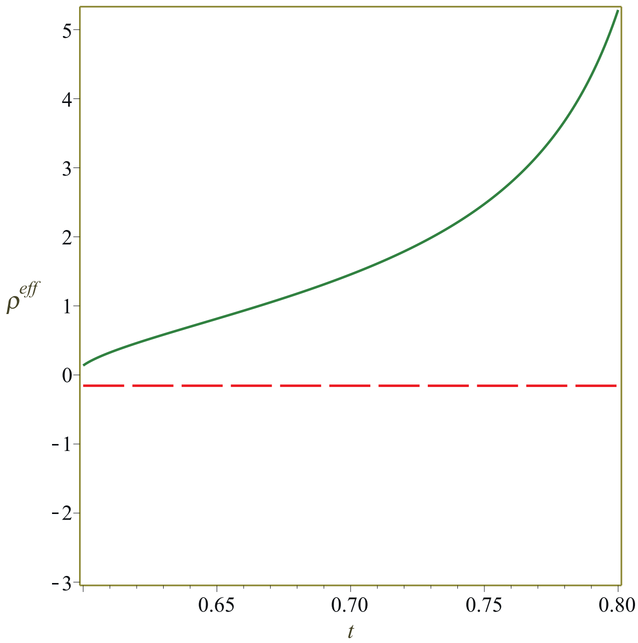

Fig. 1. Behavior of effective energy density for $\epsilon = -1$ (green line) and $\epsilon = 1$ (red line)

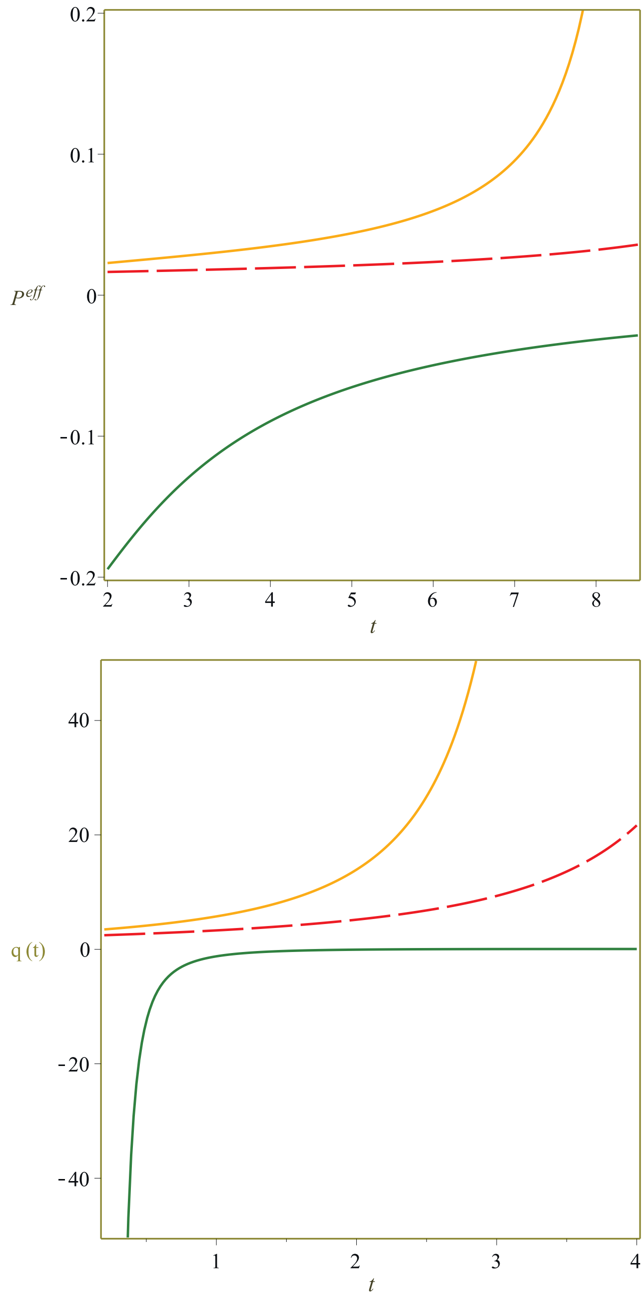

Fig. 2. Behavior of effective pressure (upper pannel) and deceleration parameter (lower pannel) for $\epsilon = -1$ (green line), $\epsilon = 1$ (red line) and $\epsilon = 0$ (orange line)

2 studies the basic formalism of EMSG. Section 3 provides a detailed study of the NS approach and derives exact cosmological solutions which are then discussed through graphs. The summary of the consequences is given in sect. 4.

2. Field Equations. We derive the field equations of the homogeneous and anisotropic spacetime in this section. The action of EMSG is expressed as [3]

$$A = \int \left(\frac{f(R,T^2)}{2\kappa^2} + L_m \right) d^4x\sqrt{-g}, \tag{1}$$

where $\kappa^2 = 1$ and $\mathcal{L}_m$ manifest the coupling constant and Lagrangian of matter, respectively. The corresponding equations of motion are obtained as

$$R_{\xi\eta}f_R + g_{\xi\eta}\Box f_R - \nabla_\xi\nabla_\eta f_R - \frac{1}{2}g_{\xi\eta}f = = T_{\xi\eta} - \Theta_{\xi\eta}f_{T^2}, \tag{2}$$

where

$$\Box = \nabla_\xi\nabla^\xi, \quad f_{T^2} = \frac{\partial f}{\partial T^2}, \quad f_R = \frac{\partial f}{\partial R}$$

and

$$\Theta_{\xi\eta} = -2L_m(T_{\xi\eta} - \frac{1}{2}g_{\xi\eta}T) - - 4\frac{\partial^2 L_m}{\partial g^{\xi\eta}\partial g^{\alpha\beta}}T^{\alpha\beta} - TT_{\xi\eta} + 2T^\alpha_\xi T_{\eta\alpha}.$$

Rearranging Eq.(2), we have

$$G_{\xi\eta} = \frac{1}{f_R}(T^{(D)}_{\xi\eta} + T_{\xi\eta}) = T^{eff}_{\xi\eta}, \tag{3}$$

where

$$T_{\xi\eta} = (\rho + p)U_\xi U_\eta + pg_{\xi\eta}$$

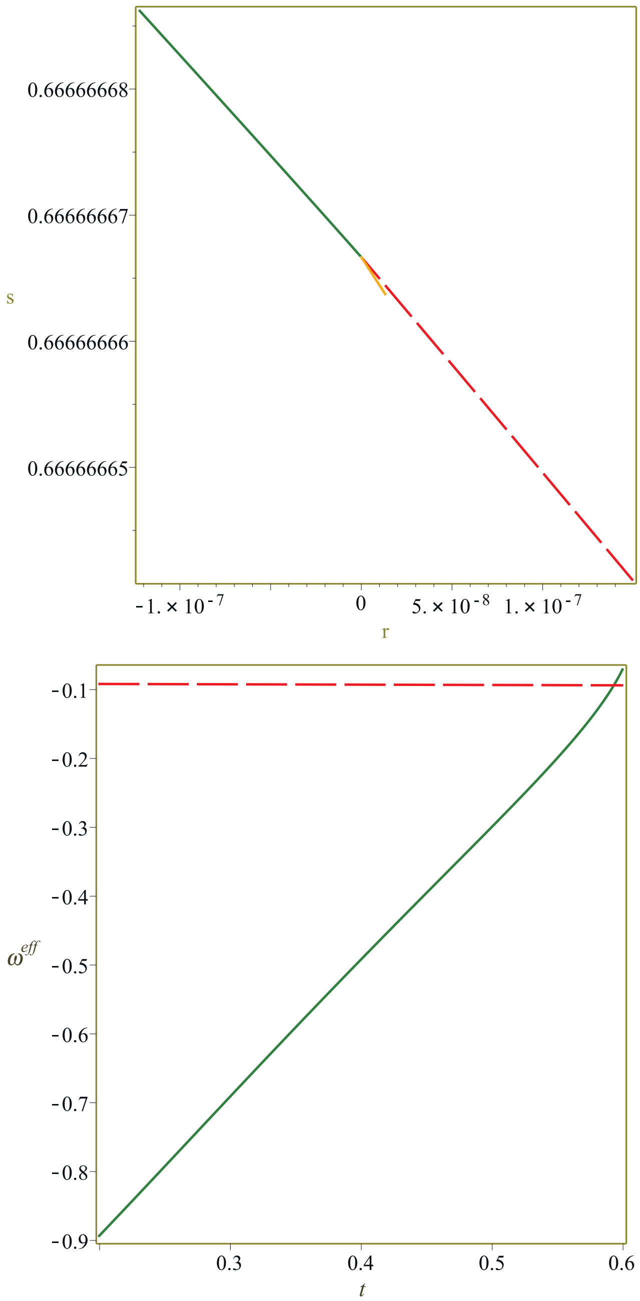

Fig. 3. Behavior of $r-s$ (upper pannel) and EoS ((lower pannel) parameters for $\epsilon=-1$ (green line), $\epsilon=1$ (red line) and $\epsilon=0$ (orange line)

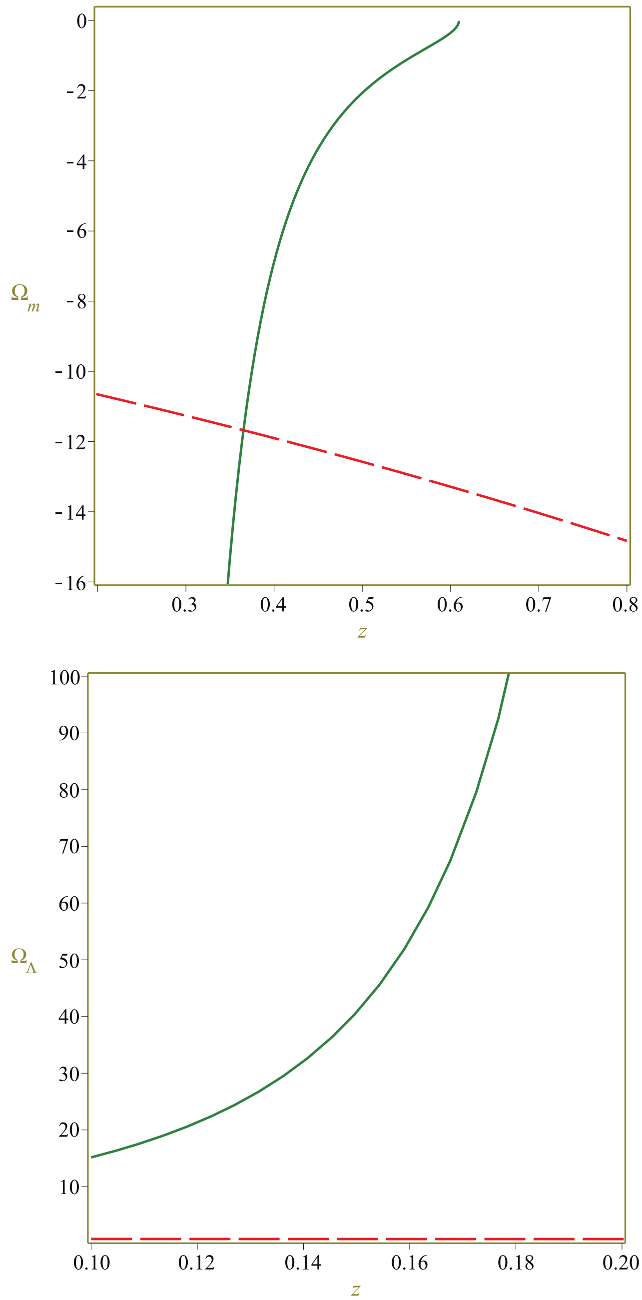

Fig. 4. Plot of Ω_m and Ω_Λ verses redshift parameter for $\epsilon=-1$ (green line) and $\epsilon=1$ (red line)

and $T^{(D)}_{\xi\eta}$ defines the modified terms of EMSG, represented as

$$T^{(D)}_{\xi\eta}=\frac{1}{2}g_{\xi\eta}(f-Rf_R)-g_{\xi\eta}\Box f_R+ \\ +\nabla_\xi\nabla_\eta f_R-\Theta_{\xi\eta}f_{T^2}. \quad (4)$$

We assume a generalized spacetime that corresponds to BT-I, BT-III and KS spacetimes as

$$ds^2=-dt^2+a^2(t)dx^2+b^2(t)(d\theta^2+\psi^2(\theta)d\phi^2), \quad (5)$$

where $\psi(\theta)=\theta,\ \sinh\theta,\ \sin\theta$ satisfying the relation

$$\frac{1}{\psi}\frac{d^2\psi}{d\theta^2}=-\epsilon.$$

For $\epsilon = 0, -1, 1$, the BT-I, BT-III and KS cosmic models are obtained. The resulting equations of motion become

$$\rho^{eff} = \frac{1}{f_R}[\rho - \frac{1}{2}f + (3p^2 + \rho^2 + 4p\rho)f_{T^2} + \\ + \epsilon b^{-2}f_R - (\dot{a}a^{-1} + 2\dot{b}b^{-1})(\dot{R}f_{RR} + \dot{T}^2 f_{RT^2}) + \\ + (\ddot{a}a^{-1} + 2\ddot{b}b^{-1} + 2\dot{a}\dot{b}a^{-1}b^{-1} + \dot{b}^2 b^{-2})f_R], \quad (6)$$

$$p^{eff} = \frac{1}{f_R}[p + \frac{1}{2}f + 2\dot{b}b^{-1}(\dot{R}f_{RR} + \dot{T}^2 f_{RT^2}) - \\ - \epsilon b^{-2}f_R + \ddot{R}f_{RR} + \ddot{T}^2 f_{RT^2} - \\ - (\ddot{a}a^{-1} + 2\ddot{b}b^{-1} + 2\dot{a}\dot{b}a^{-1}b^{-1} + \dot{b}^2 b^{-2})f_R + \\ + \dot{R}^2 f_{RRR} + \dot{T}^2 f_{RT^2T^2} + 2\dot{R}\dot{T}f_{RRT^2}], \quad (7)$$

$$p^{eff} = \frac{1}{f_R}[p + \frac{1}{2}f + (\dot{a}a^{-1} + \dot{b}b^{-1})(\dot{R}f_{RR} + \dot{T}^2 f_{RT^2}) - \\ - \epsilon b^{-2}f_R + \ddot{T}^2 f_{RT^2} + \ddot{R}f_{RR} - \\ - (\ddot{a}a^{-1} + 2\ddot{b}b^{-1} + 2\dot{a}\dot{b}a^{-1}b^{-1} + \dot{b}^2 b^{-2})f_R + \\ + \dot{R}^2 f_{RRR} + \dot{T}^2 f_{RT^2T^2} + 2\dot{R}\dot{T}f_{RRT^2}]. \quad (8)$$

Now, we apply Lagrange multiplier method to formulate the Lagrangian as

$$L = ab^2(f - Rf_R - T^2 f_{T^2} + (3p^2 + \rho^2)f_{T^2} + p) - \\ - 2a(2\dot{a}\dot{b}ba^{-1} + \dot{b}^2 - \epsilon)f_R - \\ - (2b^2\dot{a} + 4ab\dot{b})\dot{R}f_{RR} - (2b^2\dot{a} + 4ab\dot{b})\dot{T}^2 f_{RT^2}. \quad (9)$$

The fundamental properties of the system can be explained using the Hamiltonian (E) and the dynamical equations, determined as

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^i}) = 0, \quad E = \dot{q}^i(\frac{\partial L}{\partial \dot{q}^i}) - L, \quad (10)$$

where generalized coordinates are denoted by q^i. The resulting dynamical equations are

$$f - Rf_R - T^2 f_{T^2} + (3p^2 + \rho^2)f_{T^2} + p + \\ + 4\dot{b}b^{-1}(\dot{R}f_{RR} + f_{RT^2}\dot{T}^2) + a(f_{T^2}(6pp_{,a} + 2\rho\rho_{,a}) + p_{,a}) + \\ + b^{-2}(2\dot{b}^2 + 4\ddot{b}b + 2\epsilon)f_R + 2\ddot{R}f_{RR} + 2\ddot{T}^2 f_{RT^2} + 2\dot{R}^2 f_{RRR} + \\ + 2\dot{T}^2 f_{RT^2T^2} + 4\dot{R}\dot{T}^2 f_{RRT^2} = 0, \quad (11)$$

$$f - Rf_R - T^2 f_{T^2} + (3p^2 + \rho^2)f_{T^2} + (\dot{R}f_{RR} + \\ + \dot{T}^2 f_R T^2)4\dot{a}a^{-1} + p + b(f_{T^2}(6pp_{,b} + 2\rho\rho_{,b}) + p_{,b}) + \\ + 2a^{-1}b^{-1}(\ddot{a}b + \dot{a}\dot{b} + \ddot{b}a)f_R + 2\ddot{f}_R + \\ + 4\dot{b}b^{-1}(\dot{R}f_{RRT^2} + \dot{T}^2 f_{RT^2T^2}) = 0, \quad (12)$$

$$(2\ddot{a}a^{-1} - 4\ddot{b}b^{-1})f_{RR} + (3p^2 + \rho^2)f_{RT^2} - \\ - 2\epsilon b^{-1}f_{RR} - Rf_{RR} - T^2 f_{RT^2} - \\ - (4\dot{a}\dot{b}a^{-1}b^{-1} + 2\dot{b}^2 b^{-2})f_{RR} = 0, \quad (13)$$

$$(2\ddot{a}a^{-1} - 4\ddot{b}b^{-1})f_{RT^2} + (3p^2 + \rho^2)f_{T^2T^2} - \\ - 2\epsilon b^{-1}f_{RT^2} - Rf_{RT^2} - T^2 f_{T^2T^2} - \\ - (4\dot{a}\dot{b}a^{-1}b^{-1} + 2\dot{b}^2 b^{-2})f_{RT^2} = 0. \quad (14)$$

We formulate the Hamiltonian to examine the total energy of the system as

$$E = -ab^2(f - Rf_R - T^2 f_{T^2} + (3p^2 + \rho^2)f_{T^2}) - \\ - (2\dot{a}\dot{b}ba^{-1} + \dot{b}^2) \times 2af_R - \epsilon f_R - ab^2 p - \\ - (2b^2\dot{a} + 4ab\dot{b})\dot{f}_R. \quad (15)$$

The dynamical equations (11)–(14) are extremely complex due to multivariate functions and their derivatives. In the next section, we use NS technique to identify exact solutions. Although this theory is not conserved but one can obtain conserved values through NS approach, which are then used to examine the mysterious universe. As a result, this strategy is more intriguing and we adopt it in this article.

3. Noether Symmetries in EMSG. This section formulates the Noether equations for the homogenous and anisotropic universe model in EMSG. The symmetry generators are expressed as

$$Y = \lambda(t, q^i)\frac{\partial}{\partial t} + \Upsilon^j(t, q^i)\frac{\partial}{\partial q^j}, \quad i = 1, 2, 3, ..., n,$$

where $\lambda(t, a, b, R, T^2)$ and $\Upsilon^j(t, a, b, R, T^2)$ are the unknown parameters. The Lagrangian must satisfy the invariance constraint, expressed as

$$Y^{[1]}L + (D\lambda)L = D\Omega, \quad Y^{[1]} = Y + \dot{\Upsilon}^i\frac{\partial}{\partial \dot{q}^i}, \quad (16)$$

where Ω is the boundary term and

$$D = \frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i}$$

defines the total derivative. The corresponding integral integral of motion is expressed as

$$I = \Upsilon^i\frac{\partial L}{\partial \dot{q}^i} - \lambda E - \Omega. \quad (17)$$

This is a crucial component of NS that is essential for computing viable solutions and is also named as the conserved quantities.

We take the vector field (Y) with configuration space $Q = (t, a, b, R, T^2)$ to examine the generators with corresponding first integrals of Lagrangian (9) under invariance condition (16). By comparing the coefficients of Eq.(16), we have

$$2b^2\Upsilon^1_{,t}f_{RT^2}+4ab\Upsilon^2_{,t}f_{RT^2}+\Omega_{,T^2}=0, \quad (18)$$

$$2b^2\Upsilon^1_{,t}f_{RR}+4ab\Upsilon^2_{,t}f_{RR}+\Omega_{,R}=0, \quad \lambda_{,a}f_R=0, \quad (19)$$

$$b\Upsilon^1_{,T^2}f_{RR}+b\Upsilon^1_{,R}f_{RT^2}+2a\Upsilon^2_{,T^2}f_{RR}+ + 2a\Upsilon^2_{,R}f_{RT^2}=0, \quad (20)$$

$$4b\Upsilon^1_{,t}f_R+4a\Upsilon^2_{,t}f_R+4ab\Upsilon^3_{,t}f_{RR}+ + 4ab\Upsilon^4_{,t}f_{RT^2}+\Omega_{,b}=0, \quad (21)$$

$$4b\Upsilon^2_{,t}f_R+2b^2\Upsilon^3_{,t}f_{RR}+2b^2\Upsilon^4_{,t}f_{RT^2}+\Omega_{,a}=0, \quad (22)$$

$$\lambda_{,b}f_R=0,$$

$$b\Upsilon^1_{,T^2}f_{RR}+2ab\Upsilon^2_{,T^2}f_{RT^2}=0, \quad (23)$$

$$\lambda_{,R}f_{RR}=0, \quad \lambda_{,T^2}f_{RT^2}=0,$$

$$b\Upsilon^1_{,R}f_{RR}+2ab\Upsilon^2_{,R}f_{RR}=0, \quad (24)$$

$$2\Upsilon^2_{,a}f_R+b\Upsilon^3_{,a}f_{RR}+b\Upsilon^4_{,a}f_{RT^2}=0,$$

$$\Upsilon^1 f_R+a\Upsilon^3 f_{RR}+a\Upsilon^4 f_{RT^2}+2b\Upsilon^1_{,b}f_R+ + 2ab\Upsilon^4_{,b}f_{RT^2}+2ab\Upsilon^3_{,b}f_{RR}-a\lambda_{,t}f_R+ + 2a\Upsilon^2_{,b}f_R=0, \quad (25)$$

$$\lambda_{,a}f_{RR}=0, \quad \lambda_{,a}f_{RT^2}=0,$$

$$2\Upsilon^2_{,R}f_R+b\Upsilon^3_{,R}f_{RR}+b\Upsilon^4_{,R}f_{RT^2}-b\lambda_{,t}f_{RR}+ + 2\Upsilon^2 f_{RR}+b\Upsilon^1_{,a}f_{RR}+b\Upsilon^3 f_{RRR}+ + b\Upsilon^4 f_{RRT^2}+2a\Upsilon^2_{,a}f_{RR}=0, \quad (26)$$

$$\lambda_{,b}f_{RR}=0,$$

$$2\Upsilon^2 f_{RT^2}-b\lambda_{,t}f_{RT^2}+b\Upsilon^3_{,T^2}f_{RR}+ + b\Upsilon^1_{,a}f_{RT^2}+b\Upsilon^4_{,T^2}f_{RT^2}+2\Upsilon^2_{,T^2}f_R+b\Upsilon^4 f_{RT^2T^2}+ + 2a\Upsilon^2_{,a}f_{RT^2}+b\Upsilon^3 f_{RRT^2}=0, \quad (27)$$

$$\lambda_{,b}f_{RT^2}=0,$$

$$2\Upsilon^2 f_R+2b\Upsilon^3 f_{RR}+2b\Upsilon^4 f_{RT^2}+2b\Upsilon^1_{,a}f_R+2a\Upsilon^2_{,a}f_R+ + 2ab\Upsilon^3_{,a}f_{RR}+2b\Upsilon^2_{,T^2}f_R+b^2\Upsilon^3_{,b}f_{RR}+2ab\Upsilon^4_{,a}f_{RT^2}+ + b^2\Upsilon^4_{,b}f_{RT^2}-2b\lambda_{,t}f_R=0, \quad (28)$$

$$2b\Upsilon^1 f_{RR}+2a\Upsilon^2 f_{RR}+2ab\Upsilon^3 f_{RRR}+2ab\Upsilon^4 f_{RRT^2}+ + b^2\Upsilon^1_{,b}f_{RR}+2b\Upsilon^1_{,R}f_R+2ab\Upsilon^2_{,b}f_{RR}+2a\Upsilon^2_{,R}f_R+ + 2ab\Upsilon^3_{,R}f_{RR}+2ab\Upsilon^4_{,R}f_{RT^2}-2ab\lambda_{,t}f_{RR}=0, \quad (29)$$

$$\lambda_{,R}f_R=0, \quad \lambda_{,T^2}f_R=0,$$

$$2b\Upsilon^1 f_{RT^2}+2a\Upsilon^2 f_{RT^2}+2ab\Upsilon^3 f_{RRT^2}+ + 2ab\Upsilon^4 f_{RT^2T^2}+b^2\Upsilon^1_{,b}f_{RT^2}+2b\Upsilon^1_{,T^2}f_R+2ab\Upsilon^2_{,b}f_{RT^2}+ + 2a\Upsilon^2_{,T^2}f_R+2ab\Upsilon^3_{,T^2}f_{RR}+2ab\Upsilon^4_{,T^2}f_{RT^2} - 2ab\lambda_{,t}f_{RT^2}=0, \quad (30)$$

$$\lambda_{,R}f_{RT^2}=0, \quad \lambda_{,T^2}f_{RR}=0,$$

$$b^2\Upsilon^1[f-Rf_R-T^2f_{T^2}+(3p^2+\rho^2)f_{T^2}+p+ + a((6pp_{,a}+2\rho\rho_{,a})f_{T^2}+p_{,a})+2\epsilon f_R]+ + \Upsilon^2[2ab(f-Rf_R-T^2f_{T^2}+(3p^2+\rho^2)f_{T^2}+p)+ + ab^2((6pp_{,b}+2\rho\rho_{,b})f_{T^2}+p_{,b})]+(3p^2+\rho^2)f_{RT^2})+ + \Upsilon^3[-ab^2(Rf_{RR}-T^2f_{RT^2}+(3p^2+\rho^2)f_{T^2T^2})+2a\epsilon f_{RR}]+ + \Upsilon^4[-ab^2(Rf_{RT^2}-T^2f_{T^2T^2}- -(3p^2+\rho^2)f_{T^2T^2})+2a\epsilon f_{RT^2}]+ + \lambda_{,t}[ab^2(f-Rf_R-T^2f_{T^2}+2\epsilon f_R+ + (3p^2+\rho^2)f_{T^2}+p)]-\Omega_{,t}=0. \quad (31)$$

These equations help to study the dark cosmos in the context of $f(R,T^2)$. We solve the above system to obtain exact solutions for specific $f(R,T^2)$ model in the following section.

3.1. Exact Solutions. Here, we formulate the generators of NS, conserved values of the system and corresponding physical solutions. Due to the above system's complexity and nonlinearity, we assume a particular EMSG model $f(R,T^2)=R+T^2$ which minimizes the system complexity and help to examine the exact solutions. Manipulating Eqs.(18)-(31), we obtain

$$\Upsilon^1=\frac{1}{3}a\dot{F}_1(t)-2c_1a-\frac{1}{2}\frac{aF_2(t)}{b^{\frac{3}{2}}}-\frac{3}{8}\frac{F_4(t)}{\sqrt{b}}+\frac{c_2}{\sqrt{b}},$$
$$\Upsilon^2=\frac{F_2(t)}{\sqrt{b}}+(\frac{1}{3}\dot{F}_1(t)+c_1)b, \quad \lambda=F_1(t),$$
$$\Psi=-\frac{4}{3}ab^2\ddot{F}_1(t)-4a\sqrt{b}\dot{F}_2(t)+F_3(t)+\dot{F}_4(t)b^{\frac{3}{2}},$$
$$\rho=\frac{\sqrt{3c_1(-3c_1a^2\epsilon-6c_1a\epsilon-2c_3at)}}{3c_1ba}, \quad (32)$$

where $c_i(i=1,...,5)$ are integration constants with $c_1\neq 0$. The corresponding symmetry generators become

$$Y_1=-3t\frac{\partial}{\partial t}, \quad Y_2=-3a\frac{\partial}{\partial a}.$$

Substituting the value of Lagrangian (9), Hamiltonian (15) and above solutions (32) in Eq.(17), we obtain first integral as

$$I=12ab\dot{b}c_1+3\Bigg[\frac{3c_1a^2\epsilon+6c_1a\epsilon+2c_3at}{3c_1a}-$$

$$-2a\dot{b}^2-4\dot{a}\dot{b}b+2\frac{\epsilon}{b^2}\Big]c_1t-c_3t^2.$$

By comparing the coefficients of c_1 and c_3, we have

$$I_1=t^2,\quad I_2=12ab\dot{b}+3t\Big(\epsilon a-2a\dot{b}^2-4\dot{a}\dot{b}b+\frac{2\epsilon}{b^2}\Big).$$

We substitute Eqs.(32) into dynamical equations (11)-(15) and obtain the exact solution as

$$a(t)=\Big(6c_5c_3(c_5+t)^{\frac{2}{3}}-15c_1\epsilon(c_5+t)^{\frac{2}{3}}-$$
$$-4c_3t(c_5+t)^{\frac{2}{3}}+60c_4c_1\epsilon\Big)\Big(60c_1\epsilon(c_5+t)^{\frac{2}{3}}\Big)^{-1},$$

$$b(t)=\frac{1}{2}\sqrt{-6\epsilon}(c_5+t). \qquad (33)$$

To analyze this solution, we study the graphical behavior of some important cosmological parameters like deceleration and $r-s$ parameters that are the major factors in the field of cosmology. These cosmic parameters for anisotropic and homogeneous universe model are defined as

$$H=\frac{1}{3}(\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}),\quad q=-\frac{H}{H^2}-1.$$

The pair of $r-s$ parameters constructs a relationship between formulated and standard models of the universe which is used to examine the characteristics of DE, expressed as

$$r=q+2q^2-\frac{\dot{q}}{H},\quad s=\frac{r-1}{3(q-\frac{1}{2})}.$$

For $(r,s)=(1,0)$, the constructed model corresponds to ΛCDM model whereas quintessence and phantom DE eras are obtained for $s>0$ and $r<1$, respectively. The EoS parameter

$$\omega^{eff}=\frac{p^{eff}}{\rho^{eff}}$$

is a dimensionless quantity that determines the correlation between state parameters. This parameter differentiates the DE era into quintessence ($-1<\omega\leq-1/3$) and phantom ($\omega<-1$) phases.

We have considered the values of integration constants as $c_1=-2$, $c_3=-10$, $c_4=10$ and $c_5=5.7$ to analyze the graphical behavior of physical quantities. Figure 1 shows that the effective energy density is positively increasing for $\epsilon=-1$ which manifests that our universe is in the expansion phase. Figure 2 shows that the effective pressure and deceleration parameter are negative for BT-III universe model which support the current cosmic acceleration. Figure 3 determines that $r-s$ and EoS parameters describe quintessence and phantom phases of DE which represent the cosmic expansion. The obtained solutions for $\epsilon=-1$ are consistent with recent observations which indicate that this theory demonstrates expansion of the universe.

The total amount of energy density can be expressed as fractional energy density. The fractional density is defined as

$$1+\Omega_\sigma=\Omega_m+\Omega_\Lambda,$$

where

$$\Omega_m=\frac{\rho}{3H^2},\quad \Omega_\Lambda=\frac{\rho_\Lambda}{3H^2},\quad \Omega_\sigma=\frac{\sigma^2}{3H^2}.$$

The evaluation of fractional densities corresponding to ordinary matter (Ω_m) and dark energy (Ω_Λ) plays a vital role to measure the contribution of these elements in the cosmos. The densities for isotropic universe model defined as $\Omega_m+\Omega_\Lambda=1$ whereas expression equality becomes $\Omega_m+\Omega_\Lambda=1+\Omega_\sigma$ for anisotropic universe model. We analyze the behavior of fractional densities corresponding to matter and dark energy graphically at redshift scale factor where $a(t)=a_0(1+z)^{-1}$ and z is the redshift parameter. From observations of Planck data 2018, it is suggested that $\Omega_m\cong0.3111$ and $\Omega_\Lambda\cong0.6889$. According to some recent observations, there are some evidences in favor of closed universe model with fractional density $\Omega_\Lambda\cong0.01$. For $\varepsilon=-1$, the fractional density of matter indicates inconsistent behavior and the trajectory of fractional density provides $\Omega_m=0.3$ for $\varepsilon=1$ as shown in Figure 4 (left plot). In this regard, it implies consistent behavior with Planck data 2018. The right plot of Figure 4 reveals the behavior of fractional density of dark energy which shows consistent behavior with Plank data for $\varepsilon=1$ and it exhibits inconsistent behavior for $\varepsilon=-1$.

4. Final Remarks. Modified theories are assumed as the most propitious and elegant proposals to examine the dark universe due to the presence of extra higher-order geometric terms. In this paper, we have formulated exact solutions of anisotropic and homogeneous spacetimes in $f(R,T^2)$ gravity. For this reason, we have considered the NS technique to examine the exact solutions. We have formulated the Lagrangian, NS generators with conserved values in the background of EMSG. The behavior of exact solutions have been investigated through different cosmological quantities.

The main findings are summarized as follows.

1. We have established two non-zero NS generators and corresponding conserved quantities. We have obtained the exact solutions for BT-I, BT-III and KS universe models.

2. The effective energy density show accelerated and constant expansion corresponding to BT-III, BT-I and KS spacetimes, respectively (Figure 1).

3. The value of effective pressure and deceleration parameter remain negative for $\epsilon = -1$ which support the current cosmic acceleration (Figure 2).

4. The $r-s$ and EoS parameters yield quintessence and phantom DE phases which determine the rapid expansion of the universe (Figure 3).

5. In the background of BT-III universe models, the analysis of fractional density parameter of matter reveals that the EMSG is consistent with Plancks 2018 data. In case of KS universe model, this consistency is not preserved (Figure 4). We conclude that the EMSG significantly explains the cosmic journey from decelerated to accelerated epoch.

6. We find that first integrals of motion are very useful to obtain viable cosmological solutions. It is found that the considered model of EMSG supports the cosmic expansion.

The full text of this paper is published in the English version of JETP.

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