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Hall conductivity as the topological invariant in phase space in the presence of interactions and non-uniform magnetic field

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Since the discovery of the Quantum Hall Effect (QHE), there were many attempts to understand the quantization of Hall conductivity σ_H. The appearance of the universal integer values of the Hall plateaus prompts that σ_H has the topological meaning, i.e., it is related to a certain topological invariant, which is robust to the smooth modification of the system. Indeed, the seminal paper [1] shows that σ_H may be expressed through the integral of Berry curvature over the occupied electronic states. This is the so-called TKNN (Thouless, Kohmoto, Nightingale, den Nijs) invariant [2–5]. The corresponding expression is the topological invariant, i.e. it is not changed when the system is modified smoothly. However, it has been obtained for the constant magnetic fields only. Later it has been shown that in the absence of the inter-electron interactions the TKNN invariant for the intrinsic QHE (existing without external magnetic field) may be expressed through the momentum space Green's function [6, 7] (see also Chapter 21.2.1 in [8]). Recently these two results have been extended to the case of magnetic field varying as a function of coordinates. The corresponding expression for σ_H is the topological invariant in phase space expressed through the Wigner transformation of the two-point Green function [9]. The mentioned representations of σ_H through the topological invariants were derived for the non-interacting systems. It is widely believed, that in the presence of interactions the intrinsic anomalous quantum Hall effect (AQHE) conductivity is given by the expression of [6, 7], in which the non-interacting two-point Green function has been substituted by the two-point Green function with the interaction corrections [10]. In the $2+1$ D Quantum Electrodynamics this has been proved in [11]. The influence of interactions on the Hall conductivity in external magnetic field has been discussed widely in the past (see, for example [12–15] and references therein), however, this consideration has been limited by the case of the constant magnetic field. In the present work, we investigate the influence of Coulomb interactions on the QHE in the presence of the non-uniform magnetic field. On the technical side we consider the tight-binding models with the Coulomb interactions between the electrons. We will use Wigner–Weyl formalism [16–19] adapted in [20–24] to the lattice models of solid state physics combined with the ordinary perturbation theory.

Let us discuss first the system with the interactions neglected. We start from the Euclidean action of the $2+1$D tight-binding model of electrons under the action of varying magnetic field and varying electric potential, whose three-potential is A_μ

$$S = \int d\tau \sum_{\mathbf{x},\mathbf{x}'} \bar{\psi}_{\mathbf{x}'} \Big(i(i\partial_\tau - A_3(i\tau,\mathbf{x}))\delta_{\mathbf{x},\mathbf{x}'} - i\mathcal{D}_{\mathbf{x},\mathbf{x}'} \Big) \psi_{\mathbf{x}}, \quad (1)$$

where we denote the parallel transporters along the lattice vector e_i by $e^{iA_{x-e_i,x}} = e^{i\int_{x-e_i}^{x} A_\mu e_i^\mu du}$. All presented results are valid for any lattice models with the gauge invariant action of Eq. (1), and Hermitian matrix $i\mathcal{D}_{\mathbf{x},\mathbf{x}'}$. If vector potential $A_\mu(x)$ does not vary fast, i.e. if its variation on the distance of the lattice spacing may be neglected, then Wigner transformation of the two-point Green function $G_W(R,p)$ satisfies the Groenewold equation [20]

$$G_W(R,p) * Q_W(R,p) = 1,$$

in which

$$* = e^{\frac{i}{2}\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \frac{i}{2}\overleftarrow{\partial}_p \overrightarrow{\partial}_x}$$

is the star (Moyal) product, where the derivatives with the left arrow act only on the functions standing to the left from the star while the derivatives with the right arrow act only to the functions standing right to the star, while for the model with the action of Eq. (1)

$$Q_W(R,p) = \mathcal{Q}(p - A(R)).$$

Here $p = (\mathbf{p}, p_3), R = (\mathbf{R}, \tau)$. For the lattice model of a general type $\mathcal{Q}$ is a certain function specific for the

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given system. For our purposes it may be almost arbitrary. Wigner transformation of the Green function $G(p_1, p_2)$ is defined as

$$G_W(R,p) = \int_{\mathcal{M}} G(p+q/2, p-q/2) e^{iqx} dq,$$

where integral is over momentum space $\mathcal{M}$. The star product is associative: $(f * g) * h = f * (g * h)$, which allows us to write such products without brackets. The electric current density (in the absence of electric field) is given by

$$J^k(R) = \int \frac{d^3p}{(2\pi)^3} Tr G_W(R,p) \frac{\partial}{\partial p_k} Q_W(R,p). \quad (2)$$

For the convenience, we introduce the average total current $I_k = (T/S)\int J^k(R) d^3R$, in which T is temperature, while S is the area of the sample. In the following for simplicity we refer to I^k as to the total current.

The technique applied allows us to obtain the following representation for the average Hall conductivity (electric field is directed along the y axis): $\sigma_{xy} = \frac{\mathcal{N}}{2\pi}$, where $\mathcal{N}$ is the topological invariant in phase space, which is the generalization of the classical TKNN invariant [1]. Unlike the latter it is applicable to the non-homogeneous systems

$$\mathcal{N} = \frac{T}{S\, 3!\, 4\pi^2} \epsilon_{ijk} \int d^3x \int d^3p\, Tr G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k}. \quad (3)$$

This expression gives the average Hall conductivity in the presence of the non-homogeneous magnetic field and non-homogeneous electric potential, but with the interactions neglected.

It is natural to suppose also, that Eq. (3) remains valid in the presence of the inter-electron interactions. The Euclidean action is

$$S = \int d\tau \sum_{\mathbf{x},\mathbf{x}'} \Big[\bar{\psi}_{\mathbf{x}'} \Big(i(i\partial_\tau - A_3(i\tau,\mathbf{x}))\delta_{\mathbf{x},\mathbf{x}'} - i\mathcal{D}_{\mathbf{x},\mathbf{x}'} \Big) \psi_{\mathbf{x}} + + \alpha \bar{\psi}(\tau,\mathbf{x}) \psi(\tau,\mathbf{x}) \theta(y) V(\mathbf{x}-\mathbf{x}') \theta(y') \bar{\psi}(\tau,\mathbf{x}') \psi(\tau,\mathbf{x}') \Big] \quad (4)$$

with the same function $\mathcal{D}_{\mathbf{x},\mathbf{x}'}$ as above and with $A_{x,y} = \int_x^y A^\mu ds_\mu$. V is the Coulomb potential $V(\mathbf{x}) = 1/|\mathbf{x}| = 1/\sqrt{x_1^2 + x_2^2}$, for $\mathbf{x} \neq \mathbf{0}$.

The results of our calculations demonstrate, that the (averaged over the system area) Hall conductivity in the presence of inhomogeneous magnetic field, inhomogeneous electric field, and Coulomb interactions is proportional to the topological invariant in phase space of Eq. (3). In the presence of interaction one simply has to substitute to Eq. (3) the complete two-point Green function with the contribution of interactions included. In the region of analyticity in α the Hall conductivity does not depend on α at all and is still given by the same expression as without Coulomb interactions!

It would be interesting to consider the generalization of the approach of the present paper to the case, when elastic deformations are present (see, e.g., [21, 25]).

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