Spatial Kasner solution and an infinite slab with constant energy density

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The Kasner solution [1] of the Einstein equations for an empty Universe having the spatial geometry of Bianchi-I type is usually presented in the "cosmological form":

$$ds^2 = dt^2 - a_0^2 t^{2p_1} dx^2 - b_0^2 t^{2p_2} dy^2 - c_0^2 t^{2p_3} dz^2. \quad (1)$$

In the original paper by Kasner [1] the positive definite metric with the dependence on one coordinate was considered. Introducing the normal spacetime signature, one can recover not only the cosmological metric (1), but also a stationary metric that depends on one spatial coordinate:

$$ds^2 = a_0^2 (x - x_0)^{2p_1} dt^2 - dx^2 - b_0^2 (x - x_0)^{2p_2} dy^2 - - c_0^2 (x - x_0)^{2p_3} dz^2. \quad (2)$$

The metric (2) has a singularity at the hypersurface $x = x_0$, where the value x_0 is arbitrary. The Kasner indices p_1, p_2 and p_3 satisfy the relations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \quad (3)$$

A convenient parametrization of the Kasner indices was presented in paper [2]:

$$p_1 = -\frac{u}{1 + u + u^2}, \quad p_2 = \frac{1 + u}{1 + u + u^2}, \quad p_3 = \frac{u(1 + u)}{1 + u + u^2}. \quad (4)$$

It is interesting to compare the Kasner solution (2) with the external spherically symmetric Schwarzschild solution [3]. This solution has a singularity in the center of coordinates. To avoid it Schwarzschild also invented an internal solution [4] generated by a ball with constant energy density and isotropic pressure. At the boundary of the ball the pressure disappears and the external and internal solutions are matched. Then there is no singularity in the center of the ball.

The solutions of the Einstein equations in the presence of an infinite plane or an infinite slab of a finite thickness with the metric

$$ds^2 = a^2(x) dt^2 - dx^2 - b^2(x) dy^2 - c^2(x) dz^2, \quad (5)$$

were also discussed in literature [5]. When $b(x) = c(x)$, these solutions are matched with special cases of the Kasner metric (2) such as the Rindler solution [6] with $p_1 = 1$, $p_2 = p_3 = 0$ and the Weyl–Levi–Civita solution [7, 8] with $p_1 = -\frac{1}{3}$, $p_2 = p_3 = \frac{2}{3}$.

In our paper [9] we found an explicit form of two exact solutions in the spacetime with an infinite slab of thickness $2L$. In both cases pressure vanishes at the boundaries of the slab. Outside the slab these solutions are matched with the Rindler spacetime and with the Weyl–Levi–Civita spacetime. Here we describe general properties of the solutions of the Einstein equations when there is an isotropy in yz-plane, i.e., $b(x) = c(x)$, and explicitly construct a particular exact solution that differs from two solutions found in paper [9]. Besides, we discuss solutions with $b(x) \neq c(x)$, that are matched in the empty part of the space with the general Kasner solutions and not with its particular cases where $p_2 = p_3$. We are not able to write down an explicit solution of this kind, however, analyzing the corresponding differential equations we can show that such solutions do exist. Moreover, we prove that one of these empty half-spaces should have Kasner singularity.

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We introduce new functions

$$A = \frac{a'}{a},\ B = \frac{b'}{b},\ C = \frac{c'}{c}, \tag{6}$$

which permit us to write down the Einstein equations in a convenient form. We would like to find solutions of these equations inside the slab such that the pressure vanishes on its boundary. If $B = C$, then the general solution is

$$B = C = -\frac{2}{3}k \tan k(x + x_0),\ k = \frac{\sqrt{3\rho}}{2}. \tag{7}$$

We still have some freedom of choice for the function A. The simplest option is

$$A(x) = \alpha(x - L)^2 + \beta(x - L) + \gamma, \tag{8}$$

where

$$\beta = A'(L) = -\frac{1}{3}k^2 \tan^2 2kL + \frac{2}{3}\frac{k^2}{\cos^2 2kL},$$
$$\gamma = A(L) = \frac{1}{3}k \tan 2kL, \tag{9}$$

and the coefficient α is defined from the quadraric equation:

$$16\alpha^2 L^4 + \alpha(-4L - 16\beta L^3 + 8\gamma L^2) +$$
$$+ \beta + 4\beta^2 L^2 + \gamma^2 - 4\beta\gamma L - \frac{2}{3}k^2 = 0. \tag{10}$$

For $x > L$ we shall have a Weyl–Levi–Civita spacetime, while for $x < -L$ we shall have a Rindler spacetime.

Suppose now that $B(x) \neq C(x)$, and their values at the boundary $x = -L$ are also different: $B(-L) = B_0$, $C(-L) = C_0$. Then $A(-L) = A_0 = -\frac{B_0 C_0}{B_0 + C_0}$; these three numbers constitute a Kasner triplet, multiplied by a constant, and the parameter from Eq. (4) $u = \frac{C_0}{B_0}$.

At the other boundary the metric should be matched with an empty space Kasner solution for $x \geq L$

$$ds^2 = \tilde{a}_0^2(x - x_R)^{2\tilde{p}_1} dt^2 - dx^2 - \tilde{b}_0^2(x - x_R)^{2\tilde{p}_2} dy^2 -$$
$$- \tilde{c}_0^2(x - x_R)^{2\tilde{p}_3} dz^2, \tag{11}$$

with the singularity at $x = x_R$, and a triplet of the Kasner indices $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$.

In contrast to the case $B(x) = C(x)$, we cannot find an explicit particular solution of the Einstein equations in the slab that matches with two Kasner half-spaces. However, the analysis of the system of the Einstein equations with their boundary conditions permits us to show that such solutions do exist. We prove also that at least one of the Kasner empty half-spaces possesses a singularity. Thus, in contrast to the Schwarzschild solution, the Kasner type singularity cannot be avoided by introducing some simple matter distribution in the Universe.

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