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Quantum $\mathcal{R}$-matrices as universal qubit gates

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Quantum computers is quite a hot topic nowadays. They are a very promising device and method of solving lots of problems. The main problem of the quantum computers from the practical point of view is the high probability of errors. Since it has quantum nature the states are not stable and can dissolve or change. This limits the time the quantum computer can work and the size of the programs it can run. To deal with these problems the quantum correction algorithms are usually used. These algorithms imply that instead of physical qubits the logical qubits, consisting of several physical ones, are considered. The physical qubits inside of the logical qubit are regularly entangled with each other thus providing an error corrections. However, the quntum computer with such error corrections require many more physical qubits which is also a big problem at the current stage.

Another approach is to try to use qubit models where the states are more stable. One of the approaches is to make the states topological, as those are usually much harder to change. This leads to the idea of the topological quantum computer. Many of the models of this quantum computer behave under the laws of the topological 3d Chern–Simons theory. There are different models where the Chern–Simons is an effective theory which in future could provide us with the topological quantum computer [1].

The main observables, studied in the Chern–Simons theory are Wilson-loop averages, and this loops are usually thought to be related to quantum programs or algorithms. As we know these Wilson-loop averages are equal to the knot invariants. According to the Reshetikhin–Turaev approach [2], these knot invariants can be constructed from the $\mathcal{R}$-matrices. In this sense these matrices provide an elementary building blocks from which the whole knot is constructed. In quantum information theory such building blocks are called universal quantum gates. In [3] it was suggested that quantum $\mathcal{R}$-matrices can indeed be used as universal quantum gates.

In the present papers we continue these studies with the goal of studying the properties of the $\mathcal{R}$-matrices as quantum gates and how different other gates can be constructed from them using Solovay–Kitaev algorithm [4, 5]. We construct an approximation for one-qubit Hadamard and $\pi/8$ gates from fundamental $\mathcal{R}$ and Racah matrices. The generalization to larger matrices and higher representations is a work in progress.

We study the simplest topological theory which is the Chern–Simons theory. It‘s a 3d topological gauge theory with Wilson-loop averages as their most interesting observables. These Wilson-loop averages for the $SU(N)$ gauge group are equal to HOMFLY-PT polynomials. HOMFLY-PT polynomials depend on two variables A and q, which are connected to the parameters of the theory:

$$q = \exp\frac{2\pi i}{k+N}, \qquad A = q^N. \tag{1}$$

The HOMFLY-PT polynomials can be calculated as a product of quantum $\mathcal{R}$-matrices – the solutions of Yang–Baxter equation, this is called Reshetikhin–Turaev (RT) approach. One of the crucial properties of the $\mathcal{R}$-matrix is that it acts in the same way on all elements of the representation of the corresponding quantum group. Thus one can move to the $\mathcal{R}$-matrix in the space of intertwining operators, which describes how it acts on all irreducible representations in the tensor product of a pair of representations. The eigenvalues of such $\mathcal{R}$-matrices are described in a very simple way for any representation Q from the product of two representations R [2]:

$$\lambda_Q = \epsilon_Q q^{\varkappa_Q - 4\varkappa_R} A^{-|R|}. \tag{2}$$

The physically meaningful Chern–Simons theories has integer k and of course integer N. This means that both q and A are roots of unity. Thus the $\mathcal{R}$-matrix in the space of intertwining operators is unitary.

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We use the $\mathcal{R}$-matrices which appear for the two-bridge knots. Namely there appear two diagonal $\mathcal{R}$-matrices and two Racah matrices which give the non-diagonal matrices. In this letter we mainly discuss the fundamental representation. In this case these matrices are [6]:

$$T = \begin{pmatrix} q/A & \\ & -1/qA \end{pmatrix}, \quad \bar{T} = \begin{pmatrix} 1 & \\ & -A \end{pmatrix},$$
$$S = \frac{1}{\sqrt{(q+q^{-1})(A-A^{-1})}} \begin{pmatrix} \sqrt{\frac{A}{q} - \frac{q}{A}} & \sqrt{Aq - \frac{1}{Aq}} \\ \sqrt{Aq - \frac{1}{Aq}} & -\sqrt{\frac{A}{q} - \frac{q}{A}} \end{pmatrix},$$
$$\bar{S} = \begin{pmatrix} \frac{q-q^{-1}}{A-A^{-1}} & \frac{\sqrt{(Aq-\frac{1}{Aq})(\frac{A}{q}-\frac{q}{A})}}{A-A^{-1}} \\ \frac{\sqrt{(Aq-\frac{1}{Aq})(\frac{A}{q}-\frac{q}{A})}}{A-A^{-1}} & -\frac{q-q^{-1}}{A-A^{-1}} \end{pmatrix}. \quad (3)$$

We use these $\mathcal{R}$-matrices as universal quantum gates, which are a set of elementary operations, from which the quantum programs and algorithms are constructed. To use our suggest quantum gates we apply the Solovay–Kitaev theorem [4], which says that if there is a set of universal quantum gates then any program or algorithm represented by unitary matrix, can be approximated in a logarithmic time and with logarithmic number of operators. It can be proven by providing a specific algorithm constructing such an approximation, which we refer as Solovay-Kitaev algorithm [5]. The pseudocode of this algorithm reads

Algorithm 1 Solovay–Kitaev algorithm

function Solovay–Kitaev (gate U, depth n)
if $n = 0$ **then**
return Basic–Approximation (U)
else
$U_{n-1} \leftarrow$ Solovay–Kitaev(U, $n-1$)
$V, W \leftarrow$ GC-Decompose($U\, U_{n-1}^{\dagger}$)
$V_{n-1} \leftarrow$ Solovay–Kitaev(V, $n-1$)
$W_{n-1} \leftarrow$ Solovay–Kitaev(W, $n-1$)
return $V_{n-1} W_{n-1} V_{n-1}^{\dagger} W_{n-1}^{\dagger} U_{n-1}$

Fig. 1. First approximation for Hadamard gate, knots 7_4, 9_{23}, 7_4, 9_{23}, 9_4, accuracy $\epsilon = 0.0068$, $N = 2$, $k = 13$

We suggest to use the known polynomials and corresponding products of $\mathcal{R}$-matrices for knots up to some number of crossings (for example 11 or 12) as a basic approximation. Using this we construct, as an example, approximations for Hadamard (see Fig. 1) and $\pi/8$-gates E, which are the most common universal quantum gates:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}. \quad (4)$$

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